Note: Vector Space

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1 Vector Spaces and Subspaces

Let n be a non-negative interger. A *list* of *length* n [1] is an ordered collection of n elements. The elements of a list can be any entity, such as numbers, other lists, etc. A list of length n is represented by

$$(x_1, \cdots, x_n).$$

Two lists $(a_1, \cdots, a_n)$ and $(b_1, \cdots, b_n)$ are said equal if $a_i = b_i$ for $i = 1, \cdots, n$. In particular, 2-tuple is also called *ordered pair*.

Definition 1 ($\mathbb{F}^n$)**.** Let $\mathbb{F}$ be a field (for example $\mathbb{R}$ or $\mathbb{C}$). We define $\mathbb{F}^n$ to be the set of all lists of length n, where all the elements are in $\mathbb{F}$:

$$\mathbb{F}^n = \{(x_1, \cdots, x_n) : x_i \in \mathbb{F} \text{ for all } j = 1, \cdots, n\}.$$

Definition 2 (vector space)**.** A *vector space* $(V, \mathbb{F})$ is a composite object consisting of the following:

1. a field $\mathbb{F}$ of scalars;
2. a set V of objects, called *vectors*;
3. an operation, called *vector addition*, that assigns an element $u+v \in V$ to each pair of elements $u, v \in V$ such that
   (a) **commutativity.** For all $u, v \in V$, $u + v = v + u$;
   (b) **associativity of addtion.** For all $u, v, w \in V$, $(u + v) + w = u + (v + w)$;
   (c) **additive identity.** For all $v \in V$, there exists an element denoted by 0 such that $v + 0 = v$;
   (d) **additive inverse.** For all $v \in V$, there exists an element r such that $u + v = 0$;
4. an operation, called *vector multiplication*, that assigns an element $av \in V$ to each $a \in \mathbb{F}$ and each $v \in V$ such that
   (a) **multiplicative identity.** For all $v \in V$, $1v = v$;
   (b) **associativity of multiplication.** For all $a, b \in \mathbb{F}$ and for all $v \in V$, $(ab)v = a(bv)$;
   (c) **distributive property 1.** For all $a \in \mathbb{F}$ and for all $u, v \in V$, $a(u + v) = au + av$;
   (d) **distributive property 2.** For all $a, b \in \mathbb{F}$ and for all $v \in V$, $(a + b)v = av + bv$.

A vector space is usually discussed without explicitly specifying the underlying field. However, we should remember that every vector space is regarded as a vector space over a given field. Note that a vector space V is an abelian group under vector addition. Also note that the two distributive properties guarantee the closure of a vector space under two operations.

Remark 3 (vectors as an abstraction)*.* The following paragraph is quoted from the Terence Tao's lecture note on linear algebra.

[1] A list of length n is also called *n-tuple*.

We never say exactly what vectors are, only what vectors do. This is an example of *abstraction*, which appears everywhere in mathematics (but especially in algebra): the exact substance of an object is not important, only its properties and functions. (For instance, when using the number "three" in mathematics, it is unimportant whether we refer to three rocks, three sheep, or whatever; what is important is how to add, multiply, and otherwise manipulate these numbers, and what properties these operations have). This is tremendously powerful: it means that we can use a single theory (linear algebra) to deal with many very different subjects (physical vectors, population vectors in biology, portfolio vectors in finance, probability distributions in probability, functions in analysis, etc.). [A similar philosophy underlies "object-oriented programming" in computer science.] Of course, even though vector spaces can be abstract, it is often very helpful to keep concrete examples of vector spaces such as $\mathbb{R}^2$ and $\mathbb{R}^3$ handy, as they are of course much easier to visualize. For instance, even when dealing with an abstract vector space we shall often still just draw arrows in $\mathbb{R}^2$ or $\mathbb{R}^3$, mainly because our blackboards don't have all that many dimensions.

Definition 4 (subspace)**.** A subset U of a vector space V over a field $\mathbb{F}$ is called a *subspace* if U is also a vector space over $\mathbb{F}$ with the same addition and scalar multiplication defined on V.

Example 5. Let V be a vector space. The set $\{0\}$ is the smallest subspace and V itself is the largest subspace. Note that the empty set $\{\}$ is not a subspace since a vector space must at least contain an additive identity 0.

Lemma 6 (conditions for a subspace)**.** *A subset U of a vector space V is a subspace of V if and only if U satisfies*

1. ***additive identity.*** $0 \in U$*;*
2. ***closed under addition.*** $u, w \in U$ *implies* $u + w \in U$*;*
3. ***closed under scalar multiplication.*** $a \in \mathbb{F}$ *and* $u \in U$ *implies* $au \in U$*.*

Definition 7 (direct sum)**.** Suppose $U_1, \cdots, U_m$ are subspaces of V. The sum $U_1 + \cdots + U_m$ is called a direct sum if each element of $U_1 + \cdots + U_m$ can be written in only one way as a sum $u_1 + \cdots + u_m$, where each u_i is in U_i.

2 Span and Linear Independence

Definition 8 (linear combination)**.** Let V be a vector space over $\mathbb{F}$. A *linear combination* of a subset S of V is a vector of the form

$$a_1 v_1 + \cdots + a_n v_n,$$

where $a_1, \cdots, a_n \in \mathbb{F}$ and $S = \{v_1, \cdots, v_n\}$.

Definition 9 (span)**.** Let S be a nonempty subset of a vector space V. The *span* of S, denoted as $\text{span}(S)$, is the set of all the linear combinations of S. We define $\text{span}(\emptyset) = \{0\}$.

We say a set S of vectors *generates* (or *spans*) a vector space V if $\text{span}(S) = V$.

Definition 10 (finite-dimensional)**.** A vector space V is called *finite-dimensional* if there exists a set of finite number of vectors spans V. A vector space is called *infinite-dimensional* if it is not finite-dimensional.

Definition 11 (linearly independent)**.** A set S of vectors $S = \{v_1, \cdots, v_n\}$ is called *linearly independent* if the only choice of $a_1, \cdots, a_n \in \mathbb{F}$ such that

$$a_1 v_1 + \cdots + a_n v_n = 0$$

is $a_1 = \cdots = a_n = 0$. We define the empty set $\emptyset$ to be linearly independent.

A set of vectors $S = \{v_1, \cdots, v_n\}$ is called *linearly dependent* if it is not linearly independent. In other words, there exists a list of scalars $a_1, \cdots, a_n \in \mathbb{F}$, not all zero, such that $a_1 v_1 + \cdots + a_n v_n = 0$.

3 Bases and Dimension

Definition 12 (basis)**.** A *basis* of a vector space V is a set of vectors that is linearly independent and spans V.

Proposition 13 (criterion for basis)**.** *Let $S = \{v_1, \cdots, v_n\}$ be a subset of a vector space V. Then, S is a basis of V if and only if every $v \in V$ can be uniquely written as*

$$a_1 v_1 + \cdots + a_n v_n,$$

where $a_1, \cdots, a_n \in \mathbb{F}$.

Proposition 14. *Let V be a finite-dimensional vector space. Then, every basis of V contains the same number of vectors.*

Proposition 14 states that the length of a basis of the vector space V is an intrinsic property of V, regardless of the choice of the basis. We call this intrinsic property of vector spaces as dimension, which is formally defined as follow.

Definition 15 (dimension)**.** The *dimension* of a finite-dimensional vector space V, denoted as $\dim(V)$, is the number of vectors of any basis of V.

Example 16. The vector space $\{0\}$ has dimension zero.